

## THE COX RING OF A DEL PEZZO SURFACE

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ABSTRACT. Let  $X_r$  be a smooth Del Pezzo surface obtained from  $\mathbb{P}^2$  by blowing up  $r \leq 8$  points in general position. It is well known that for  $r \in \{3, 4, 5, 6, 7, 8\}$  the Picard group  $\text{Pic}(X_r)$  contains a canonical root system  $R_r \in \{A_2 \times A_1, A_4, D_5, E_6, E_7, E_8\}$ . We prove some general properties of the Cox ring of  $X_r$  ( $r \geq 4$ ) and show its similarity to the homogeneous coordinate ring of the orbit of the highest weight vector in some irreducible representation of the algebraic group  $G$  associated with the root system  $R_r$ .

## 1. INTRODUCTION

Let  $X$  be a projective algebraic variety over a field  $\mathbb{k}$ . Assume that the Picard group  $\text{Pic}(X)$  is a finitely generated abelian group. Consider the vector space

$$\Gamma(X) := \bigoplus_{[D] \in \text{Pic}(X)} H^0(X, \mathcal{O}(D)).$$

One wants to make it an  $\mathbb{k}$ -algebra which is graded by the monoid of effective classes in  $\text{Pic}(X)$  such that the algebra structure will be compatible with the natural bilinear map

$$b_{D_1, D_2} : H^0(X, \mathcal{O}(D_1)) \times H^0(X, \mathcal{O}(D_2)) \rightarrow H^0(X, \mathcal{O}(D_1 + D_2)).$$

However, there exist some problems in the realization of this idea. First of all there is no any natural isomorphism between  $H^0(X, \mathcal{O}(D))$  and  $H^0(X, \mathcal{O}(D'))$  if  $[D] = [D']$ . There exists only a canonical bijection between the linear systems  $|D| \cong |D'|$  (where  $|D|$  is the projectivization of the  $\mathbb{k}$ -vector space  $H^0(X, \mathcal{O}(D))$ ). As a consequence, the bilinear map  $b_{D_1, D_2}$  depends not only on the classes  $[D_1], [D_2], [D_1 + D_2] \in \text{Pic}(X)$ , but also on their particular representatives. One can easily see that only the morphism

$$s_{[D_1], [D_2]} : |D_1| \times |D_2| \rightarrow |D_1 + D_2|$$

of the product of two projective spaces  $|D_1| \times |D_2|$  to another projective space  $|D_1 + D_2|$  is well-defined. For this reason, it is much more natural to consider the graded set of projective spaces

$$|\Gamma(X)| := \bigsqcup_{[D] \in \text{Pic}(X)} |D|$$

together with all possible morphisms  $s_{[D_1], [D_2]}$  any two effective classes  $[D_1], [D_2] \in \text{Pic}(X)$ .

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Inspired by the paper of Cox on the homogeneous ring of a toric variety [Cox], Hu and Keel [H-K] suggested a definition of a *Cox ring*

$$\text{Cox}(X) = R(X, L_1, \dots, L_r) := \bigoplus_{(m_1, \dots, m_r) \in \mathbb{Z}^r} H^0(X, \mathcal{O}(m_1 L_1 + \dots + m_r L_r))$$

which uses a choice of some  $\mathbb{Z}$ -basis  $L_1, \dots, L_r$  in  $\text{Pic}(X)$  (e.g. if  $\text{Pic}(X) \cong \mathbb{Z}^r$  is a free abelian group). Using such a  $\mathbb{Z}$ -basis, one obtains a particular representative for each class in  $\text{Pic}(X)$  together with a well-defined multiplication so  $R(X, L_1, \dots, L_r)$  becomes a well-defined  $\mathbb{k}$ -algebra. If  $L'_1, \dots, L'_r$  is another  $\mathbb{Z}$ -basis of  $\text{Pic}(X)$ , then the corresponding Cox algebra  $R(X, L'_1, \dots, L'_r)$  is isomorphic to  $R(X, L_1, \dots, L_r)$ . Unfortunately, we can not expect to choose a  $\mathbb{Z}$ -basis of  $\text{Pic}(X)$  in a natural canonical way. More often one can choose in a natural way some effective divisors  $D_1, \dots, D_n$  on  $X$  such that  $\text{Pic}(X)$  is generated by  $[D_1], \dots, [D_n]$ . If we set

$$U := X \setminus (D_1 \cup \dots \cup D_n)$$

and assume that  $X$  is smooth, then  $\text{Pic}(U) = 0$  and we obtain the exact sequence

$$1 \rightarrow \mathbb{k}^* \rightarrow \mathbb{k}[U]^* \rightarrow \bigoplus_{i=1}^n \mathbb{Z}[D_i] \rightarrow \text{Pic}(X) \rightarrow 0.$$

Choosing a  $\mathbb{k}$ -rational point  $p$  in  $U$ , we can split the monomorphism  $\mathbb{k}^* \rightarrow \mathbb{k}[U]^*$ , so that one has an isomorphism

$$\mathbb{k}[U]^* \cong \mathbb{k}^* \oplus G,$$

where  $G \subset \mathbb{k}[U]^*$  is a free abelian group of rank  $n - r$ . The choice of a  $\mathbb{k}$ -rational point  $p \in U$  allows to give another approach to the graded space  $\Gamma(X)$  and to the Cox algebra:

**Definition 1.1.** Let  $X, U, p, D_1, \dots, D_n$  be as above. We consider the graded  $\mathbb{k}$ -algebra

$$\Gamma(X, U, p) := \bigoplus_{(m_1, \dots, m_n) \in \mathbb{Z}^n} H^0(X, \mathcal{O}(m_1 D_1 + \dots + m_n D_n))$$

and define

$$\text{Cox}(X, U, p) := \Gamma(X, U, p)_G$$

as the quotient of the  $\Gamma(X, U, p)$  modulo the ideal generated by

$$\{x - gx \mid x \in \Gamma(X, U, p), g \in G\}.$$

Since  $\text{Pic}(X) \cong \mathbb{Z}^n/G$ , we obtain a natural  $\text{Pic}(X)$ -grading on  $\text{Cox}(X, U, p)$ .

We expect that the algebra  $\text{Cox}(X, U, p)$  can be applied to some arithmetic questions on  $\mathbb{k}$ -rational points in  $U \subset X$ .

*Remark 1.2.* The above definition of the ring  $\text{Cox}(X, U, p)$  depends on the choice of an open subset  $U \subset X$  and a  $\mathbb{k}$ -rational point  $p \in U$ . A similar idea was used by Colliot-Thélène and Sansuc in [C-S] for constructing universal torsors and deriving explicit equations for them. The lack of a canonical construction is precisely what makes descending the universal torsor an interesting problem. Some applications of the universal torsor for Del Pezzo surfaces of degree 5 was considered by Skorobogatov in [S1] (see also [S2]). Recently, Hassett and Tschinkel have investigated the Cox rings and the universal torsors for some interesting special cubic surfaces [H-T].

*Remark 1.3.* If  $X$  is a smooth projective toric variety and  $U \subset X$  is the open dense torus orbit, then the choice of a point  $p \in U$  defines an isomorphism of  $U$  with the algebraic torus  $T$ , so that the subgroup  $G \subset \mathbb{k}[U]^*$  can be identified with the character group of  $T$ . In this way, one can show that  $\text{Cox}(X, U, p)$  is isomorphic to a polynomial ring in  $n$  variables ( $n$  is the number of irreducible components of  $X \setminus U$ , cf. [Cox]).

*Remark 1.4.* The field of fractions of the ring  $\text{Cox}(X, U, p)$  is a pure transcendental extension of degree  $r$  of the field of rational functions on  $X$ . Therefore,  $\dim \text{Spec } \Gamma(X, U, p) = \dim X + r$ , if  $\Gamma(X, U, p)$  is a finitely generated  $\mathbb{k}$ -algebra.

Let  $X_r$  be a smooth Del Pezzo surface obtained from  $\mathbb{P}^2$  by blow-up of  $r \leq 8$  points in general position. It is well known that for  $r \in \{3, 4, 5, 6, 7, 8\}$  the Picard group  $\text{Pic}(X_r)$  contains a canonical root system  $R_r \in \{A_2 \times A_1, A_4, D_5, E_6, E_7, E_8\}$ . Moreover, the natural embedding  $\text{Pic}(X_{r-1}) \hookrightarrow \text{Pic}(X_r)$  induces the inclusion of root systems  $R_{r-1} \hookrightarrow R_r$ . If  $G(R_r)$  is a connected algebraic group corresponding to the root system  $R_r$ , then the embedding  $R_{r-1} \hookrightarrow R_r$  defines a maximal parabolic subgroup  $P(R_{r-1}) \subset G(R_r)$  [Hm]. We expect that for  $r \geq 4$  there should be some relation between a Del Pezzo surface  $X_r$  and the GIT-quotient of the homogeneous space  $G(R_r)/P(R_{r-1})$  modulo the action of a maximal torus  $T_r$  of  $G(R_r)$ .

Our starting observation is the well-known isomorphism  $X_4 \cong G(3, 5) // T_4$  which follows from an isomorphism between the homogeneous coordinate ring of the Grassmannian  $G(3, 5) = G(A_4)/P(A_2 \times A_1) \subset \mathbb{P}^9$  and the Cox ring of  $X_4$  (see 4.1). Another proof of this fact follows from the identification of  $X_4$  with the moduli space  $\overline{M}_{0,5}$  of stable rational curves with 5 marked points [K].

In this paper, we start an investigation of the Cox ring of Del Pezzo surfaces  $X_r$  ( $r \geq 4$ ). It is natural to choose the classes of all exceptional curves  $E_1, \dots, E_{N_r} \subset X_r$  as a generating set for the Picard group  $\text{Pic}(X_r)$ . There is a natural  $\mathbb{Z}_{\geq 0}$ -grading on  $\text{Pic}(X_r)$  defined by the intersection with the anticanonical divisor  $-K$ .

We prove some general properties of the Cox rings of a Del Pezzo surface  $X_r$  ( $r \geq 4$ ) and show their similarity to the homogeneous coordinate ring of  $G(R_r)/P(R_{r-1})$ . We remark that the homogeneous space  $G(R_r)/P(R_{r-1})$  can be interpreted as the orbit of the highest weight vector in some natural irreducible representation of  $G(R_r)$ .

*Remark 1.5.* Some other connections between Del Pezzo surfaces and the corresponding algebraic groups were considered also by Friedman and Morgan in [F-M]. A similar topic was considered by Leung in [Le].

In this paper, we show that the Cox ring of a Del Pezzo surface  $X_r$  is generated by elements of degree 1. This implies that the homogeneous coordinate ring of  $G(R_r)/P(R_{r-1})$  is naturally graded by the monoid of effective divisor classes on the surface  $X_r$  (the same monoid defines the multigrading of the Cox ring of  $X_r$ ). Moreover, we obtain some results of the quadratic relations between the generators of the Cox ring of  $X_r$ .

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## 2. DEL PEZZO SURFACES

Let us summarize briefly some well-known classical results on Del Pezzo surfaces which can be found in [Ma, Dem, Na].

One says that  $r$  ( $r \leq 8$ ) points  $p_1, \dots, p_r$  in  $\mathbb{P}^2$  are in *general position* if there are no 3 points on a line, no 6 points on a conic ( $r \geq 6$ ) and a cubic having seven points and one of them double does not have the eighth one ( $r = 8$ ).

Denote by  $X_r$  ( $r \geq 3$ ) the Del Pezzo surfaces obtained from  $\mathbb{P}^2$  by blowing up of  $r$  points  $p_1, \dots, p_r$  in general position. If  $\pi : X_r \rightarrow \mathbb{P}^2$  the corresponding projective morphism, then the Picard group  $\text{Pic}(X_r) \cong \mathbb{Z}^{r+1}$  contains a  $\mathbb{Z}$ -basis  $l_i$ , ( $0 \leq i \leq r$ ),  $l_0 = [\pi^*\mathcal{O}(1)]$  and  $l_i := [\pi^{-1}(p_i)]$ ,  $i = 1, \dots, r$ . The intersection form  $(*, *)$  on  $\text{Pic}(X_r)$  is determined in the chosen basis by the diagonal matrix:  $(l_0, l_0) = 1$ ,  $(l_i, l_i) = -1$  for  $i \geq 1$ ,  $(l_i, l_j) = 0$  for  $i \neq j$ . The anticanonical class of  $X_r$  equals  $-K = 3l_0 - l_1 - \dots - l_r$ . The number  $d := (K, K) = 9 - r$  is called the *degree* of  $X_r$ . The anticanonical system  $|-K|$  of a Del Pezzo surface  $X_r$  is very ample if  $r \leq 6$ , it determines a two-fold covering of  $\mathbb{P}^2$  if  $r = 7$ , and it has one base point, determining a rational map to  $\mathbb{P}^1$  if  $r = 8$ . Smooth rational curves  $E \subset X_r$  such that  $(E, E) = -1$  and  $(E, -K) = 1$  are called *exceptional curves*.

**Theorem 2.1.** [Ma] *The exceptional curves on  $X_r$  are the following:*

- (1) *blown-up points  $p_1, \dots, p_r$ ;*
- (2) *lines through pairs of points  $p_i, p_j$ ;*
- (3) *conics through 5 points from  $\{p_1, \dots, p_r\}$  ( $r \geq 5$ );*
- (4) *cubics, containing 7 points and 1 of them double ( $r \geq 7$ );*
- (5) *quartics, containing 8 points and 3 of them double ( $r = 8$ );*
- (6) *quintics, containing 8 of point and 6 of them double ( $r = 8$ );*
- (7) *sextics, containing 8 of those points, 7 of them double and 1 triple ( $r = 8$ ).*

The number  $N_r$  of exceptional curves on  $X_r$  is given by the following table:

$r$	3	4	5	6	7	8
$N_r$	6	10	16	27	56	240

The root system  $R_r \subset \text{Pic}(X_r)$  is defined as

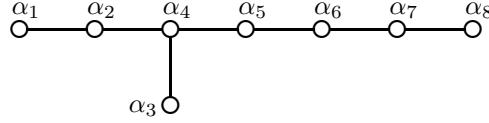
$$R_r := \{\alpha \in \text{Pic}(X_r) : (\alpha, \alpha) = -2, (\alpha, -K) = 0\}.$$

It is easy to show that  $R_r$  is exactly the set of all classes  $\alpha = [E_i] - [E_j]$  where  $E_i$  and  $E_j$  are two exceptional curves on  $X_r$  such that  $E_i \cap E_j = \emptyset$ .

The corresponding Weyl group  $W_r$  is generated by the reflections  $\sigma : x \mapsto x + (x, \alpha)\alpha$  for  $\alpha \in R_r$ . There are so called *simple roots*  $\alpha_1, \dots, \alpha_r$  such that the corresponding reflexions  $\sigma_1, \dots, \sigma_r$  form a minimal generating subset of  $W_r$ . The set of simple roots can be chosen as

$$\begin{aligned} \alpha_1 &= l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = l_0 - l_1 - l_2 - l_3, \\ \alpha_i &= l_{i-1} - l_i, \quad i \geq 4. \end{aligned}$$

The blow up morphism  $X_r \rightarrow X_{r-1}$  determines an isometric embedding of the Picard lattices  $\text{Pic}(X_{r-1}) \hookrightarrow \text{Pic}(X_r)$ . This induces the embeddings for root systems, simple roots and Weyl groups  $W_r$ . For  $r \geq 3$ , the Dynkin diagram of  $R_r$  can be considered as the subgraph on the vertices  $\alpha_i$  ( $i \leq r$ ) of the following graph:



In particular, we obtain  $R_3 = A_2 \times A_1$ ,  $R_4 = A_4$ ,  $R_5 = D_5$ ,  $R_6 = E_6$ ,  $R_7 = E_7$ ,  $R_8 = E_8$ .

Denote by  $\varpi_1, \dots, \varpi_r$  the dual basis to the  $\mathbb{Z}$ -basis  $-\alpha_1, \dots, -\alpha_r$ . Each  $\varpi_i$  is the highest weight of an irreducible representation of  $G(R_r)$  which is called a *fundamental representation*. We shall denote by  $V(\varpi)$  the representation space of  $G(R_r)$  with the highest weight  $\varpi$ .

**Definition 2.2.** A dominant weight  $\varpi$  is called *minuscule* if all weights of  $V(\varpi)$  are nonzero and the  $W_r$ -orbit of the highest weight vector is a  $\mathbb{k}$ -basis of  $V(\varpi)$  [G/P-I]. A dominant weight  $\varpi$  is called *quasiminuscule* [G/P-III], if all nonzero weights of  $V(\varpi)$  have multiplicity 1 and form an  $W_r$ -orbit of  $\varpi$  (the zero weight of  $V(\varpi)$  may have some positive multiplicity).

One can see from the explicit description of the root systems  $R_r$  that  $\varpi_r$  is minuscule for  $3 \leq r \leq 7$ , and  $\varpi_8$  is quasiminuscule.

The dimension  $d_r$  of the irreducible representation  $V(\varpi_r)$  of  $G(R_r)$  is given by the following table:

$r$	4	5	6	7	8
$d_r$	10	16	27	56	248

We will need the following statement:

**Proposition 2.3.** Let  $D$  be a divisor on a Del Pezzo surface  $X_r$  ( $2 \leq r \leq 8$ ) such that  $(D, E) \geq 0$  for every exceptional curve  $E \subset X_r$ . Then the following statements hold:

- (i) the linear system  $|D|$  has no base points on any exceptional curve  $E \subset X_r$ ;
- (ii) if  $r \leq 7$ , then the linear system  $|D|$  has no base points on  $X_r$  at all.

*Proof.* Induction on  $r$ . If  $r = 2$ , then there exists exactly 3 exceptional curves  $E_0, E_1, E_2$ , whose classes in the standard basis are  $l_0 - l_1 - l_2, l_1, l_2$ . Moreover  $[E_0], [E_1]$  and  $[E_2]$  form a basis of the Picard lattice  $\text{Pic}(X_2)$ . The dual basis w.r.t. the intersection form is  $l_0, l_0 - l_1, l_0 - l_2$ . Therefore the above conditions on  $D$  imply that

$$[D] = n_0 l_0 + n_1 (l_0 - l_1) + n_2 (l_0 - l_2), \quad n_0, n_1, n_2 \in \mathbb{Z}_{\geq 0}$$

So it is sufficient to check that the linear systems with the classes  $l_0, l_0 - l_1, l_0 - l_2$  have no base points. The latter immediately follows from the fact that the first system defines the birational morphism  $X_2 \rightarrow \mathbb{P}^2$  contracting  $E_1$  and  $E_2$ , the second and third linear systems define conic bundle fibrations over  $\mathbb{P}^1$ .

For  $r > 2$ , we consider a second induction on  $\deg D = (D, -K)$ .

If there is an exceptional curve  $E \subset X_r$  with  $(D, E) = 0$ , then the invertible sheaf  $\mathcal{O}(D)$  is the inverse image of an invertible sheaf  $\mathcal{O}(D')$  on the Del Pezzo surface  $X_{r-1}$  obtained by the contraction of  $E$ . Since the pull-back of any exceptional curve on  $X_{r-1}$  under the birational morphism  $\pi_E : X_r \rightarrow X_{r-1}$  is again an exceptional curve on  $X_r$ ,

we obtain that  $D'$  satisfy all conditions of the proposition on  $X_{r-1}$ . By the induction assumption ( $r-1 \leq 7$ ),  $|D'|$  has no base points on  $X_{r-1}$ . Therefore,  $|D| = |\pi_E^* D'|$  has no base points on  $X_r$ .

If there is no exceptional curve  $E \subset X_r$  with  $(D, E) = 0$ , then we denote by  $m$  the minimal intersection number  $(D, E)$  where  $E$  runs over all exceptional curves. Since we have  $(E, -K) = 1$  for all exceptional curves, the divisor  $D' := D + mK$  has nonnegative intersections with all exceptional curves and there exists an exceptional curve  $E \subset X_r$  with  $(D', E) = 0$ . Since  $\deg D' = (D', -K) = (D, -K) - m(K, K) < (D, -K) = \deg D$ , by the induction assumption, we obtain that  $|D'|$  is base point free. If  $r \leq 7$ , then the anticanonical linear system  $|-K|$  has no base points. Therefore,  $|D| = |D' + m(-K)|$  is also base point free. In the case  $r = 8$ ,  $|-K|$  does have a base point  $p \in X_8$ . However,  $p$  cannot lie on an exceptional curve  $E$ , because the short exact sequence

$$0 \rightarrow H^0(X_8, \mathcal{O}(-K - E)) \rightarrow H^0(X_8, \mathcal{O}(-K)) \rightarrow H^0(E, \mathcal{O}(-K)|_E) \rightarrow 0$$

induces an isomorphism  $H^0(X_8, \mathcal{O}(-K)) \cong H^0(E, \mathcal{O}(-K)|_E)$  (since  $\deg(-K - E) = 0$  and  $H^0(X_8, \mathcal{O}(-K - E)) = 0$ ).  $\square$

### 3. GENERATORS OF $\text{Cox}(X_r)$

Let  $\{E_1, \dots, E_{N_r}\}$  be the set of all exceptional curves on a Del Pezzo surface  $X_r$ . We choose a  $\mathbb{k}$ -rational point  $p \in U := X_r \setminus (\bigcup_{i=1}^{N_r} E_i)$  and denote the ring  $\text{Cox}(X_r, U, p)$  (see 1.1) simply by  $\text{Cox}(X_r)$ .

The ring

$$\text{Cox}(X_r) = \bigoplus_{[D] \in M_{\text{eff}}(X_r)} \text{Cox}(X_r)^{[D]}$$

is graded by the semigroup  $M_{\text{eff}}(X_r) \subset \text{Pic}(X_r)$  of classes  $[D]$  of effective divisors  $D$  on  $X_r$ . There is a coarser grading on  $\text{Cox}(X_r)$  given by

$$\text{Cox}(X_r)^d := \bigoplus_{\deg [D]=d} \text{Cox}(X_r)^{[D]},$$

where  $\deg [D] := (D, -K)$ .

**Proposition 3.1.** *The graded ring  $\text{Cox}(X_3)$  is isomorphic to a polynomial ring in 6 variables  $\mathbb{k}[x_1, \dots, x_6]$ , where  $x_i$  are sections defining all 6 exceptional curves on  $X_3$ .*

*Proof.* The Del Pezzo surface  $X_3$  is a toric variety which can be described as the blow-up of 3 torus invariant points  $(1:0:0)$ ,  $(0:1:0)$  and  $(0:0:1)$  in  $\mathbb{P}^2$ . So we can apply a general result of Cox on toric varieties [Cox] (see also 1.3).  $\square$

**Theorem 3.2.** *For  $3 \leq r \leq 8$ , the ring  $\text{Cox}(X_r)$  is generated by elements of degree 1. If  $r \leq 7$ , then the generators of  $\text{Cox}(X_r)$  are global sections of invertible sheaves defining the exceptional curves. If  $r = 8$ , then we should add to the above set of generators two linearly independent global sections of the anticanonical sheaf on  $X_8$ .*

*Proof.* Induction on  $r$ . The case  $r = 3$  is settled by the previous proposition.

For  $r > 3$  we choose an effective divisor  $D$  on  $X_r$ . We call a section  $s \in H^0(X_r, \mathcal{O}(D))$  a *distinguished global section* if its support is contained in the union of exceptional curves

of  $X_r$  ( $r \leq 7$ ), or if its support is contained in the union of exceptional curves of  $X_8$  and some anticanonical curves on  $X_8$ . Our purpose is to show that the vector space  $H^0(X_r, \mathcal{O}(D))$  is spanned by all distinguished global sections.

This will be proved by induction on  $\deg D := (D, -K) > 0$ .

We consider several cases:

- If there exists an exceptional curve  $E$  such that  $(D, E) < 0$ , then  $H^0(X_r, \mathcal{O}(D)|_E) = 0$  and it follows from the exact sequence

$$H^1(X_r, \mathcal{O}(D)|_E) \rightarrow H^0(X_r, \mathcal{O}(D - E)) \rightarrow H^0(X_r, \mathcal{O}(D)) \rightarrow 0$$

that the multiplication by a non-zero distinguished global section of  $\mathcal{O}(E)$  induces an epimorphism  $H^0(X_r, \mathcal{O}(D - E)) \rightarrow H^0(X_r, \mathcal{O}(D))$ . Since  $\deg(D - E) = \deg D - 1$ , using the induction assumption for  $D' = D - E$ , we obtain the required statement for  $D$ .

- If there exists an exceptional curve  $E$  such that  $(D, E) = 0$ , then  $\mathcal{O}(D)$  is the inverse image of a sheaf  $\mathcal{O}(D')$  on the Del Pezzo surface  $X_{r-1}$  obtained by the contraction of  $E$ . Therefore we have an isomorphism  $H^0(X_r, \mathcal{O}(D)) \cong H^0(X_{r-1}, \mathcal{O}(D'))$  and, by the induction assumption for  $r - 1$ , we obtain the required statement for  $D$ , because distinguished global sections of  $\mathcal{O}(D')$  lift to distinguished global sections of  $\mathcal{O}(D)$ .
- If  $D = -K$ , (or, equivalently, if  $(D, E) = 1$  for every exceptional curve  $E$ ), then  $\mathcal{O}(D)|_E$  is isomorphic to  $\mathcal{O}_E(1)$  and we have  $H^1(X_r, \mathcal{O}(D)|_E) = 0$  together with the exact sequence

$$0 \rightarrow H^0(X_r, \mathcal{O}(D - E)) \rightarrow H^0(X_r, \mathcal{O}(D)) \rightarrow H^0(X_r, \mathcal{O}(D)|_E) \rightarrow 0,$$

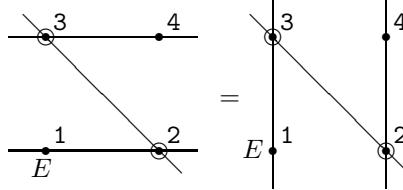
where  $H^0(X_r, \mathcal{O}(D)|_E)$  is 2-dimensional. Since  $\deg(D - E) = \deg D - 1 < \deg$ , we can apply the induction assumption for  $D' = D - E$ . It remains show that there exists two linearly independent distinguished global sections of  $\mathcal{O}(D)$  such that their restriction to  $E$  are two linearly independent global sections of  $\mathcal{O}(D)|_E$ . We describe these two distinguished sections explicitly for each value of  $r \in \{4, 5, 6, 7, 8\}$ . Without loss of generality we can assume that  $[E] = l_1$ .

If  $r = 4$ , then we write the anticanonical class  $-K = 3l_0 - l_1 - \cdots - l_4$  in the following two ways:

$$\begin{aligned} -K &= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_2 - l_3) + l_2 + l_3 \\ &= (l_0 - l_1 - l_3) + (l_0 - l_2 - l_4) + (l_0 - l_2 - l_3) + l_2 + l_3. \end{aligned}$$

These two decompositions of  $-K$  determine two distinguished global sections of  $\mathcal{O}(-K)$  with support on 5 exceptional curves. The projections of these sections under the morphism  $X_4 \rightarrow \mathbb{P}^2$  are shown below in Figure 1.

The restriction of the first section to  $E$  vanishes at the intersection point  $q_1$  of  $E$  with the exceptional curve with the class  $l_0 - l_1 - l_2$ . The restriction of the second section to  $E$  vanishes at the intersection point  $q_2$  of  $E$  with the exceptional curve with the class  $l_0 - l_1 - l_3$ . It is clear that  $q_1 \neq q_2$ . So the distinguished anticanonical sections are linearly independent.

FIGURE 1. Two distinguished anticanonical classes for  $r = 4$ .

If  $r = 5$ , then we write the anticanonical class as

$$\begin{aligned} -K &= 3l_0 - l_1 - \cdots - l_5 \\ &= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_4 - l_5) + l_4 \\ &= (l_0 - l_1 - l_5) + (l_0 - l_2 - l_3) + (l_0 - l_3 - l_4) + l_3. \end{aligned}$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of  $E$  with the exceptional curves belonging to the classes  $l_0 - l_1 - l_2$  and  $l_0 - l_1 - l_5$ .

If  $r = 6$ , then we write the anticanonical class as

$$\begin{aligned} -K &= 3l_0 - l_1 - \cdots - l_6 \\ &= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_5 - l_6) \\ &= (l_0 - l_1 - l_6) + (l_0 - l_5 - l_4) + (l_0 - l_3 - l_2). \end{aligned}$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of  $E$  with the exceptional curves belonging to the classes  $l_0 - l_1 - l_2$  and  $l_0 - l_1 - l_6$ .

If  $r = 7$ , then we write the anticanonical class as

$$\begin{aligned} -K &= 3l_0 - l_1 - \cdots - l_7 \\ &= (2l_0 - l_1 - l_2 - l_3 - l_4 - l_5) + (l_0 - l_6 - l_7) \\ &= (2l_0 - l_7 - l_6 - l_5 - l_4 - l_3) + (l_0 - l_2 - l_1). \end{aligned}$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of  $E$  with the exceptional curves belonging to the classes  $2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$  and  $l_0 - l_2 - l_1$ .

If  $r = 8$ , then  $\deg D - E = 0$ . Therefore,  $H^0(X_8, \mathcal{O}(D - E)) = 0$  (see the proof of 2.3) and we have an isomorphism

$$H^0(X_8, \mathcal{O}(D)) \cong H^0(X_8, \mathcal{O}(D)|_E).$$

So  $H^0(X_8, \mathcal{O}(D)|_E)$  is generated by the restrictions of the anticanonical sections and we're done.

- If  $(D, E) \geq 1$  for all exceptional curves  $E$  and  $D \neq -K$ , then we denote by  $m$  the minimum of the numbers  $(D, E)$  for all exceptional curves. Let  $E_0$  be an exceptional curve such that  $(D, E_0) = m \geq 1$ . We define  $D' = D - E_0$  and  $D'' := D + mK$ . By 2.3,  $|D'|$  and  $|D''|$  have no base points (if  $r \leq 7$ ). In particular,  $D''$  is represented by an effective divisor. Since  $\deg D'' = \deg D - m(K, K) < \deg D$ ,  $D''$  can be seen as zero of a distinguished global section  $s \in H^0(X_r, \mathcal{O}(D + mK))$  whose support does not contain the exceptional curve  $E_0$  (if  $r \leq 8$ ). We have the short exact sequence

$$0 \rightarrow H^0(X_r, \mathcal{O}(D')) \rightarrow H^0(X_r, \mathcal{O}(D)) \rightarrow H^0(X_r, \mathcal{O}(D)|_{E_0}) \rightarrow 0.$$

By the induction assumption, the space  $H^0(X_r, \mathcal{O}(D'))$  is generated by distinguished global sections. It remains to show that there exist distinguished global sections of  $\mathcal{O}(D)$  such that their restriction to  $E_0$  generate the space  $H^0(X_r, \mathcal{O}(D)|_{E_0})$ . Since  $(-mK, E_0) = (D, E_0) = m$ , the space  $H^0(X_r, \mathcal{O}(D)|_{E_0})$  is isomorphic to  $H^0(X_r, \mathcal{O}(-mK)|_{E_0})$ . Since  $(D'', E_0) = 0$  the distinguished global section  $s \in H^0(X_r, \mathcal{O}(D + mK))$  is nonzero at any point of  $E_0$ . Therefore the multiplication by the distinguished global section  $s$  defines a homomorphism

$$H^0(X_r, \mathcal{O}(-mK)) \rightarrow H^0(X_r, \mathcal{O}(D))$$

whose restriction to  $E_0$  is an isomorphism

$$H^0(X_r, \mathcal{O}(-mK)|_{E_0}) \cong H^0(X_r, \mathcal{O}(D)|_{E_0}).$$

Therefore, it is enough to show that restrictions of the distinguished global sections of  $\mathcal{O}(-mK)$  to  $E_0$  generate the space  $H^0(X_r, \mathcal{O}(-mK)|_{E_0})$ . Our previous considerations have shown this for  $m = 1$ . The general case  $m \geq 1$  follows now immediately from the fact that the homomorphism  $H^0(X_r, \mathcal{O}(-K)) \rightarrow H^0(E_0, \mathcal{O}_{E_0}(1))$  is surjective and the space  $H^0(E_0, \mathcal{O}_{E_0}(m))$  is spanned by tensor products of  $m$  elements from  $H^0(E_0, \mathcal{O}_{E_0}(1))$ .  $\square$

**Corollary 3.3.** *The semigroup  $M_{\text{eff}}(X_r) \subset \text{Pic}(X_r)$  of classes of effective divisors on a Del Pezzo surfaces  $X_r$  ( $2 \leq r \leq X_r$ ) is generated by elements of degree 1. These elements are exactly the classes of exceptional curves if  $r \leq 7$  and the classes of exceptional curves together with the anticanonical class for  $r = 8$ .*

**Proposition 3.4.** *If  $D$  is an effective divisor of degree  $\geq 2$  on  $X_8$ , then the vector space  $H^0(X_8, \mathcal{O}(D))$  is spanned by distinguished global sections of  $\mathcal{O}(D)$  with supports only on exceptional curves.*

*Proof.* By 3.2 and 3.3, it is sufficient to check the statement for  $D = -2K$  and for  $D = -K + E$  for any exceptional curve. The latter case immediately follows from 3.2, because  $D = -K + E$  is the pull back of the anticanonical sheaf on  $X_7$  obtained by the contraction of  $E$ . In the case  $D = -2K$ , we obtain 120 distinguished global sections of  $\mathcal{O}(D)$  from 120 pairs of exceptional curves  $E_i, E_j$  such that  $(E_i, E_j) = 3$ :

$$-2K = 6l_0 - 2l_1 - \dots - 2l_8 = l_1 + (6l_0 - 3l_1 - 2l_2 - \dots - 2l_8).$$

It is well-known (see e.g. [Dem]) that  $X_8$  can be realized as a hypersurface of degree 6 in the weighted projective space  $\mathbb{P}(3, 2, 1, 1)$ . In particular, the linear system  $|-2K|$  defines

a double covering of  $X_8$  over a singular quadratic cone  $\mathcal{Q} \cong \mathbb{P}(2,1,1) \subset \mathbb{P}^3$ . The single singular point  $p \in \mathcal{Q}$  is the image of the base-point  $b \in X_8$  of  $|-K|$  on  $X_8$ . Let  $C \subset \mathcal{Q}$  be the ramification locus ( $C$  is a curve of degree 6 in  $\mathbb{P}(2,1,1)$ ). Then 120 pairs of exceptional curves  $E_i, E_j$  on  $X_8$  such that  $[E_i] + [E_j] = 2[-K]$  one-to-one correspond to conics in  $\mathbb{P}(2,1,1)$  which are 3-tangent to the ramification curve  $C$ . Since every such conic in  $\mathcal{Q}$  is uniquely determined as  $\mathcal{Q} \cap H$  for some plane  $H \subset \mathbb{P}^3$ . Therefore, the distinguished sections in  $H^0(X_8, \mathcal{O}(-2K))$  can be identified (up to a scalar multiple) with the above planes  $H \subset \mathbb{P}^3$ . It remains to show that all these 120 planes  $H$  cannot pass through the some common point  $x \in \mathbb{P}^3$  for a generic choice of the sextic  $C \subset \mathbb{P}(2,1,1)$ . The later can be checked by standard dimension arguments.  $\square$

*Remark 3.5.* Since  $H^0(X_r, \mathcal{O}(E))$  is 1-dimensional for each exceptional curve  $E \subset X_r$ , we can choose a nonzero section  $x_E \in H^0(X_r, \mathcal{O}(E))$  which is determined up to multiplication by a nonzero scalar. Therefore the affine algebraic variety  $\mathbb{A}(X_r) := \text{Spec } \text{Cox}(X_r)$  is embedded into the affine space  $\mathbb{A}^{N_r}$  on which the maximal torus  $T_r \subset G(R_r)$  acts in a canonical way such that the space  $\mathbb{A}^{N_r}$  can be identified with the representation space  $V(\varpi_r)$  of the algebraic group  $G(R_r)$  (if  $r \leq 7$ ). In the case  $r = 8$ , all 240 exceptional curves on  $X_8$  can be similarly identified with all non-zero weights of the adjoint representation of  $G(E_8)$  in  $V(\varpi_8)$ . The space  $V(\varpi_8)$  contains the weight-0 subspace of dimension 8, but the ring  $\text{Cox}(X_r)$  has only 2-dimensional space of anticanonical sections. Thus, we cannot identify the degree-1 homogeneous component of  $\text{Cox}(X_8)$  with the representation space  $V(\varpi_8)$  of  $G(E_8)$ .

Since the kernel of the surjective homomorphism

$$\deg : \text{Pic}(X_r) \rightarrow \mathbb{Z},$$

can be identified with the character group  $\mathfrak{X}(T_r)$  of a maximal torus  $T_r \subset G(R_r)$  and the torus  $T_r$  acts on the homogeneous space  $G(R_r)/P(R_{r-1})$  embedded into the projective space  $\mathbb{P}V(\varpi_r)$  we obtain a natural  $\text{Pic}(X_r)$ -grading of the homogeneous coordinate ring of the projective variety  $G(R_r)/P(R_{r-1})$ .

**Theorem 3.6.** *Let  $\lambda$  be an element in  $\text{Pic}(X_r)$ . The weight- $\lambda$  subspace in the homogeneous coordinate ring of the projective variety  $G(R_r)/P(R_{r-1})$  is nonzero if and only if  $\lambda$  is represented by an effective divisor on  $X_r$  (i.e.,  $\lambda \in M_{\text{eff}}(X_r)$ ).*

*Proof.* It is known that the projective variety  $G(R_r)/P(R_{r-1})$  is arithmetically normal and Cohen-Macaulay [D-L, G/P-V]. In particular, the homogeneous coordinate ring of  $G(R_r)/P(R_{r-1})$  is generated by elements of degree 1. Therefore, the weight- $\lambda$  subspace in the coordinate ring is nonzero if and only if  $\lambda$  is a nonnegative integral linear combination of  $\text{Pic}(X_r)$ -weights having positive multiplicity in  $V(\varpi_r)$ . By 3.3 and 3.5, the latter is equivalent to  $\lambda \in M_{\text{eff}}(X_r)$ .

#### 4. QUADRATIC RELATIONS IN $\text{Cox}(X_r)$

Let us denote  $\mathbb{P}(X_r) := \text{Proj } \text{Cox}(X_r)$ . If  $4 \leq r \leq 7$ , then the projective variety  $\mathbb{P}(X_r)$  is canonically embedded into the projective space  $\mathbb{P}^{N_r-1}$  ( $N_r$  is the number of exceptional curves on  $X_r$ ). The affine variety  $\mathbb{A}(X_r) \subset \mathbb{A}^{N_r}$  is the affine cone over  $\mathbb{P}(X_r)$ .

**Proposition 4.1.** *The ring  $\text{Cox}(X_4)$  is isomorphic to the subring of all  $3 \times 3$ -minors of a generic  $3 \times 5$ -matrix. In particular, the projective variety  $\mathbb{P}(X_4) \subset \mathbb{P}^9$  is isomorphic to the Plücker embedding of the Grassmannian  $Gr(3, 5)$ .*

*Proof.* In order to describe the multiplication in  $\text{Cox}(X_4)$ , one needs to choose a basis in  $\text{Pic}(X_4)$ .

Let  $x : y : z$  be the homogeneous coordinates on  $\mathbb{P}^2$ . We choose the basis  $l_0, \dots, l_4$ , as in Section 2, i.e.,  $l_0$  is the preimage of the line  $z = 0$  at infinity,  $l_1, l_2, l_3, l_4$  are classes of the exceptional fibers over 4 points  $p_1, \dots, p_4 \in \mathbb{P}^2$ . We identify the representatives of each class in  $\text{Pic}(X_4)$  with the subsheaves  $\mathcal{O}(\sum_{i=0}^4 m_i l_i)$  of the constant sheaf  $\mathcal{K}(X_4)$  of rational functions on  $X_4$ . Then the ring multiplication in  $\text{Cox}(X_4)$  is just the multiplication of the corresponding rational functions in  $\mathcal{K}(X_4)$ .

Let  $(x_i : y_i : z_i)$  be the coordinates of the blown-up point  $p_i \in \mathbb{P}^2$  ( $i = 1, \dots, 4$ ). Consider the  $3 \times 5$ -matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x/z \\ y_1 & y_2 & y_3 & y_4 & y/z \\ z_1 & z_2 & z_3 & z_4 & 1 \end{pmatrix}.$$

For any 3-element subset  $I = \{i, j, k\} \subset \{1, \dots, 5\}$ , we denote by  $M_I$  the maximal minor of  $M$  consisting of the columns with numbers in  $I$  taken in the natural order.

We choose the rational functions in  $\mathcal{K}(X_4)$  representing the generators  $x_E$  of  $\text{Cox}(X_4)$  as follows:

$$\begin{aligned} x_{l_1} &:= M_{\{2,3,4\}}, \quad x_{l_1} := M_{\{1,3,4\}}, \quad x_{l_3} := M_{\{1,2,4\}}, \quad x_{l_1} := M_{\{1,2,3\}}, \\ x_{l_0-l_i-l_j} &:= M_{\{i,j,5\}}, \quad 1 \leq i < j \leq 4. \end{aligned}$$

All these functions are non-zero because the points  $p_1, \dots, p_4$  are in general position.

It is known that the generators of the homogeneous coordinate ring of  $G(3, 5)$  are naturally identified with the maximal minors of a generic  $3 \times 5$ -matrix. Consider the homomorphism  $\varphi$  of the homogeneous coordinate ring of  $G(3, 5)$  to  $\text{Cox}(X_4)$ , which sends these generic minors into the corresponding minors of the matrix  $M$  above. Since  $\text{Cox}(X_4)$  is generated by  $\{x_E\}$ , this homomorphism is surjective. By 3.6,  $\varphi$  respects the  $\text{Pic}(X_r)$ -grading (in particular,  $\varphi$  respects the  $\mathbb{Z}_{\geq 0}$ -grading as well). The surjectivity of  $\varphi$  induces a closed embedding of  $\mathbb{P}(X_4)$  into  $G(3, 5)$ . Since both varieties are irreducible of dimension 6 (see 1.4), we obtain an isomorphism of  $\mathbb{P}(X_4)$  and  $G(3, 5)$  as subvarieties of  $\mathbb{P}^9$ . Therefore  $\varphi$  is an isomorphism of the homogeneous coordinate ring of  $G(3, 5)$  and  $\text{Cox}(X_4)$ . In particular,  $\text{Cox}(X_4)$  is defined by 5 quadratic Plücker relations. One of these relations is

$$M_{\{1,2,5\}}M_{\{3,4,5\}} - M_{\{1,3,5\}}M_{\{2,4,5\}} + M_{\{1,4,5\}}M_{\{2,3,5\}} = 0.$$

□

The article [G/P-I] describes a  $\mathbb{k}$ -basis for the homogeneous coordinate ring of  $G/P$  in the case, when  $P$  is a maximal parabolic subgroup containing a Borel subgroup  $B$  such that the fundamental weight  $\varpi$  corresponding to  $P$  is minuscule (see 2.2). It also shows that this ring is always defined by quadratic relations.

A way to write explicitly the quadratic relations for the orbit of the highest weight vector for any representation of a semisimple Lie group is given in [Li]. A more geometric approach to these quadratic equations is contained in the proof of Theorem 1.1 in [L-T]:

**Proposition 4.2.** *The orbit  $G/P_{\varpi}$  of the highest weight vector in the projective space  $\mathbb{P}V(\varpi)$  is the intersection of the second Veronese embedding of  $\mathbb{P}V(\varpi)$  with the subrepresentation  $V(2\varpi)$  of the symmetric square  $S^2V(\varpi)$ . Moreover, these quadratic relations generate the ideal of  $G/P_{\varpi} \subset \mathbb{P}V(\varpi)$ .*

We expect that the following general statement is true:

**Conjecture 4.3.** *The ideal of relations between the degree-1 generators of  $\text{Cox}(X_r)$  is generated by quadrics for  $4 \leq r \leq 8$ .*

For any exceptional curve  $E \subset X$ , we consider the open chart  $U_E \subset \mathbb{P}^{N_r-1}$  defined by the condition  $x_E \neq 0$ . Thus, we obtain an open covering of  $\mathbb{P}(X_r)$  by  $N_r$  affine subsets  $U_E \cap \mathbb{P}(X_r)$ .

**Proposition 4.4.** *Let  $X_{r-1}$  the Del Pezzo surface obtained by the contraction of  $E$  on  $X_r$ . Then there exist a natural isomorphism*

$$U_E \cap \mathbb{P}(X_r) \cong \mathbb{A}(X_{r-1}).$$

*Proof.* Let  $\pi : X_r \rightarrow X_{r-1}$  be the contraction of  $E$ . Then we obtain the ring homomorphism  $\pi^* : \text{Cox}(X_{r-1}) \rightarrow \text{Cox}(X_r)$ . We shall show that the localization  $\text{Cox}(X_r)_{x_E}$  of the ring  $\text{Cox}(X_r)$  by the element  $x_E$  can be identified with the Laurent polynomial extension of  $\pi^* \text{Cox}(X_{r-1})$  by  $x_E$ , i.e. there exist a ring isomorphism

$$\text{Cox}(X_r)_{x_E} \cong \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

For simplicity, we assume that  $[E] = l_r$  and  $\{l_0, \dots, l_{r-1}\}$  is the pull-back of the standard basis in  $\text{Pic}(X_{r-1})$ . We remark that any divisor class

$$[D] = m_r l_r + \sum_{i=0}^{r-1} m_i l_i \in \text{Pic}(X_r)$$

is uniquely represented as sum  $m_r [E] + [D']$  where  $[D'] = \sum_{i=0}^{r-1} m_i l_i \in \pi^*(\text{Pic}(X_{r-1}))$ . Using  $\pi^*$ , we identify two fields of rational functions  $\mathcal{K}(X_{r-1})$  and  $\mathcal{K}(X_r)$ . This identification allows us to consider the vector space

$$H^0(X_r, \mathcal{O}(m_r l_r + \sum_{i=0}^{r-1} m_i l_i))$$

as a subspace of

$$H^0(X_{r-1}, \mathcal{O}(\sum_{i=0}^{r-1} m_i l_i)) x_E^{m_r} \subset \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

For fixed integers  $m_0, m_1, \dots, m_{r-1}$ , the embedding of the vector spaces

$$H^0(X_r, \mathcal{O}(m_r l_r + \sum_{i=0}^{r-1} m_i l_i)) \hookrightarrow H^0(X_{r-1}, \mathcal{O}(\sum_{i=0}^{r-1} m_i l_i)) x_E^{m_r}$$

is an isomorphism for sufficiently large  $m_r$ . Moreover, this embedding of vector spaces respects the multiplications in  $\text{Cox}(X_r)$  and in  $\pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}]$ . Thus, we obtain an embedding of rings

$$\text{Cox}(X_r) \hookrightarrow \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

On the other hand, it is clear that  $\pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}]$  is a subring of the localization  $\text{Cox}(X_r)_{x_E}$ . Thus, we get an isomorphism

$$\text{Cox}(X_r)_{x_E} \cong \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

Now we remark that the coordinate ring of the affine variety  $U_E \cap \mathbb{P}(X_r)$  is degree-0 component of  $\text{Cox}(X_r)_{x_E}$ . By the above isomorphism, this component is isomorphic to  $\text{Cox}(X_{r-1})$ .  $\square$

**Corollary 4.5.** *The singular locus of the algebraic varieties  $\mathbb{P}(X_r)$  and  $\mathbb{A}(X_r)$  has codimension 7.*

*Proof.* Since  $\mathbb{A}(X_3) \cong \mathbb{A}^6$ , we obtain that  $\mathbb{P}(X_4)$  is a smooth variety covered by 10 affine charts which are isomorphic to  $\mathbb{A}^6$ . Using the isomorphism  $\mathbb{P}(X_4) \cong G(3, 5)$  (see 4.1 ), we obtain that  $\mathbb{A}(X_4)$  has an isolated singularity at 0. Therefore, the singular locus of  $\mathbb{P}(X_5)$  consists of 16 isolated points. The singular locus of  $\mathbb{P}(X_6)$  is 1-dimensional and the singular locus of  $\mathbb{P}(X_7)$  is 2-dimensional etc.  $\square$

**Definition 4.6.** A divisor class  $[D]$  is called a *ruling* if it can be written as a sum of two classes of exceptional curves  $[E_i] + [E_j]$  such that  $(E_i, E_j) = 1$ , or , equivalently, if  $D$  satisfies the conditions  $(D, D) = 0$ ,  $(D, -K) = 2$ . The invertible sheaf corresponding to a ruling determines a conic bundle morphism  $X_r \rightarrow \mathbb{P}^1$ .

*Remark 4.7.* Lemma 5.3 of [F-M] says that the Weyl group acts transitively on rulings.

Each ruling  $[D]$  can be represented by  $r - 1$  different ways as a sum of two classes of exceptional curves corresponding to degenerate fibers of the conic bundle  $X_r \rightarrow \mathbb{P}^1$ . Thus, we obtain  $r - 1$  distinguished sections in the 2-dimensional space  $H^0(X_r, \mathcal{O}(D))$ . If  $r \geq 4$ , then for each ruling  $[D]$ , we obtain in this way  $r - 3$  linearly independent quadratic relations between generators of  $\text{Cox}(X_r)$ .

*Remark 4.8.* We note that  $\text{Pic}(X_4)$  has exactly 5 rulings. Each such a ruling defines a Plücker quadric (see the proof of 4.1).

We cannot expect in general that all quadratic relation among generators are coming from rulings. However, the following statement is true:

**Theorem 4.9.** *For  $4 \leq r \leq 6$ , the ring  $\text{Cox}(X_r)$  is defined by the radical of the ideal generated by the quadratic relations corresponding to rulings.*

*Proof.* Let  $Z_r \subset \mathbb{A}^{N_r}$  is the affine subvariety defined by the quadratic relations coming from rulings. We want to show that  $Z_r = \mathbb{A}(X_r)$  ( $4 \leq r \leq 6$ ).

For  $r = 4$ , the statement follows from 4.8.

Obviously, the zero  $0 \in \mathbb{A}^{N_r}$  is common point of  $Z_r$  and  $\mathbb{A}(X_r)$  for all  $r$ . Consider the affine open coverings of  $Z_r \setminus \{0\}$  and  $\mathbb{A}(X_r) \setminus \{0\}$  defined by affine open subsets  $x_E \neq 0$ , where  $E$  runs over all exceptional curves of  $X_r$ . Using the induction on  $r$  and Proposition

4.4, we want to show that  $Z_r \cap U_E = \mathbb{A}(X_r) \cap U_E$  for each exceptional curve. For this purpose, it is important to remark that the affine coordinate ring of  $Z_r \cap U_E$  is generated by all elements  $x_F/x_E$  such that  $(E, F) = 0$ . For  $r = 5, 6$ , the last property follows from the fact that if  $(E, E') > 0$  for two exceptional curves  $E, E'$  on  $X_r$ , then  $(E, E') = 1$ , i.e.,  $[E] + [E']$  is a ruling and there exists a ruling quadratic relation

$$x_E x_{E'} = \sum_i a_i X_{E_i} X_{E'_i},$$

where all exceptional curves  $E_i, E'_i$  do not intersect  $E$ . The last property shows that

$$Z_r \cap U_E \cong Z_{r-1} \times (\mathbb{A}^1 \setminus \{0\}).$$

It follows from the proof of 4.4 that

$$\mathbb{A}(X_r) \cap U_E \cong \mathbb{A}(X_{r-1}) \times (\mathbb{A}^1 \setminus \{0\}).$$

By induction, we have the equality  $Z_{r-1} = \mathbb{A}(X_{r-1})$ . This implies the equality  $Z_r \cap U_E = \mathbb{A}(X_r) \cap U_E$  for each exceptional curve. Thus,  $Z_r = \mathbb{A}(X_r)$ .  $\square$

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